Immersed Boundary Method implementation of the turbulent boundary layer at the ice shelf/ocean interface

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Abstract

This document presents an algorithm for computing the fluid forcing required to impose the boundary conditions associate with the turbulent planetary boundary layer near the ice shelf/ocean interface. The implementation is indended for use in an Immersed Boundary Method (IBM) implemented in the Parallel Ocean Program (POP), and is intended for use in the near future in coupling the physics between POP and Glimmer: The Community Ice Sheet Model (Glimmer-CISM). The analytic solutions for the variation of the velocity and the active tracer quantities (temperature and salinity) in the boundary layer.

1. Nomenclature

Symbol		**Description**
a	$=$	$(1-\eta*)/(\eta_*\xi_N)$
a_T		salinity coefficient in linearized freezing temperature formula
$b(x,y)$		the (positive) height field representing the ice/ocean interface
b_T		temperature offset in linear freezing point relaiton
C_D		non-dimensional drag coefficient
$c_{p,i}$		specific heat capacity of ice
$c_{p,o}$		specific heat capacity of ocean
c_T		pressure coefficient in linearized freezing temperature formula
f		magnitude of Corilois parameter (always positive)
g		acceleration of gravity
$H(x,y)$		the (positive) depth of the bathymetry below sea level
i	$=$	$\sqrt{-1}$
K		eddy viscosity
K_T		turbulent diffusivity of heat
K_S		turbulent diffusivity of salt
k	$=$	0.4, von Karman's constant
L	$=$	$\rho_o u_*^3/\left[gk\left(\gamma_S\left\langle u_z'S'\right\rangle_0-\gamma_T\left\langle u_z'T'\right\rangle_0\right)\right]$, Obukhov length
L_f		latent heat of fusion

Symbol		Description
n		distance from interface (positive into ice)
n_0		surface roughness
n_{bl}	$=$	$-ku_*\eta_*/f$, edge of the boundary layer
n_{ref}		reference distance from interface
n_{sl}		$-u_*\eta_*^2\xi_N/f$, edge of the surface layer
p_0		pressure at interface (depth dependent)
R_c	$=$	0.2, critical Richardson number
S		salinity
S_0		salinity at interface
T		temperature
T_0		temperature at interface
$\hat{\boldsymbol{u}}$	$=$	$\eta_*(\boldsymbol{u}_t - \boldsymbol{u}_{t,\infty})/\boldsymbol{u}_*$, complex non-dimensional tangential velocity deviation
$\hat{\boldsymbol{u}}_0$		complex non-dimensional tangential surface velocity deviation
$\boldsymbol{u}_i$		the velocity of the ice at the interface (supplied by the ice sheet model)
u_n		normal velocity
$u_{n,\text{melt}}$		melt rate in ocean
$u_{n,\text{melt},i}$		melt rate of ice
$\boldsymbol{u}_t$		complex tangential velocity
$\boldsymbol{u}_{t,\infty}$		complex tangential velocity outside the boundary layer
$\boldsymbol{u}_*$		complex surface friction velocity
χ	$=$	± 1, positive for northern, negative for southern hemisphere
δ	$=$	$(\chi i/k\xi_N)^{1/2}$, complex attenuation coefficient
η_*	$=$	$(1+\xi_N\mu_*/R_c)^{-1/2}$, stability parameter
κ_i^T		molecular diffusivity heat in ice
κ_o^S		molecular diffusivity salt in ocean water
κ_o^T		molecular diffusivity heat in ocean water
λ		one of $\{T,S\}$
μ_*	$=$	$u_*/(fL)$
ν		molecular viscosity of ocean water
$\Phi_{T,S}$		non-dimensional change of $\{T,S\}$ over boundary layer
$\Phi_{T,S,\text{ref}}$		non-dimensional reference value for change of $\{T,S\}$
Φ_{turb}		non-dimensional change of $\{T,S\}$ due to turbulence
$\Phi_{T,S}^{\text{mol}}$		non-dimensional change of $\{T,S\}$ due to molecular processes
ρ_i		density of ice at interface
ρ_o		density of ocean water at interface
ξ_N	$=$	0.052, dimensionless universal constant
ζ	$=$	$fn/u_*\eta_*$, non-dimensional distance from interface (positive into ice)
ζ_0		non-dimensional surface roughness
ζ_{ref}		reference non-dimensional distance from interface

2. Forcing Points in the Immersed Boundary Method

The Immersed Boundary Method (IBM) used to represent the boundary between the ice shelf and the ocean in the Parallel Ocean Program (POP) can apply forcing at grid points adjacent to the boundary location that are either just exterior to the fluid (ghost points) or just interior to the fluid (band points). Ghost points have the desirable property that the forcing does not directly modify the fluid in the "real" portion of the computational domain, but only the "fictitious" portion that is simulated within the solid body (the ice shelf). For this reason, I have opted to use the ghost point forcing, exterior to the "real" fluid domain in my IBM. For each ghost point, I first use an interpolation method to compute the value of the fluid properties at an image point that lies in the real fluid. Then I linearly extrapolate values for the fluid properties at each ghost point using the image and boundary values.

3. Tangential velocity boundary layer solution

McPhee (1981) proposed an analytic solution for the mean tangential velocity (mean in the sense of the Reynolds average) within the turbulent boundary layer below the ice/ocean interface. The velocity solution is broken into two parts, one for the sublayer in which the eddy viscosity varies with distance from the interface and where viscous and roughness effects play a role, and one for the outer layer in which the eddy viscosity can be considered to be constant. The velocity solution is represented as a complex number, where the real part is the x component and the imaginary part is the y component. The non-dimensional form of the solution is (McPhee, 1981, Eq. (17)):

$$\hat{\boldsymbol{u}} = \begin{cases} -i\chi\delta e^{\delta\zeta} & \zeta \leq -\xi_N \\ -i\chi\delta e^{-\delta\xi_N} - \frac{\eta_*}{k}\left[\ln\frac{|\zeta|}{\xi_N} + (\delta - a)(\zeta + \xi_N) - \frac{a}{2}\delta(\zeta^2 - \xi_N^2)\right], & \zeta > -\xi_N \end{cases}, \tag{1}$$

where $k = 0.4$ and $\xi_N = 0.052$ are universal constants, where $\chi = \pm 1$ (positive in the northern hemisphere, negative in the southern), $\delta = (\chi i/k\xi_N)^{1/2}$ where $\hat{\boldsymbol{u}} = \eta_*(\boldsymbol{u}_t - \boldsymbol{u}_{t,\infty})/\boldsymbol{u}_*$ and $\zeta = fn/u_*\eta_*$, and where $\boldsymbol{u}_*$ is the friction velocity (with magnitude equal to the square root of the magnitude of the kinematic stress) at the interface, $\boldsymbol{u}_{t,\infty}$ is the velocity outside the boundary layer, f is the (positive) local Coriolis parameter, and n is the distance from the interface in the direction normal to the interface (negative into the ocean, so that if the interface is horizontal, $n = z$, the usual height above sea level). The effects of buoyancy are parameterized in terms of L, the Obukhov length scale, or non-dimensional parameters μ_*, η_* and a involving this length:

$$L \equiv \frac{\rho_o u_*^3}{gk\left(\gamma_S \left\langle u_z' S' \right\rangle_0 - \gamma_T \left\langle u_z' T' \right\rangle_0\right)}, \tag{2}$$

$$\mu_* \equiv \frac{u_*}{fL}, \tag{3}$$

$$\eta_* \equiv \left(1 + \frac{\xi_N \mu_*}{R_c}\right)^{-1/2}, \tag{4}$$

$$a \equiv \frac{1-\eta_*}{\eta_*\xi_N}, \tag{5}$$

where γ_S and γ_T are the expansion coefficients for salinity and temperature, respectively, and where $R_c \approx 0.2$ is the critical Richardson number. The Reynolds averaged vertical salinity flux $\left\langle u_z' S' \right\rangle_0$ and temperature flux $\left\langle u_z' T' \right\rangle_0$ will be discussed in Sec. 4, where methods are given for computing these values in terms of the bulk fluid properties.

Values for $\boldsymbol{u}_{t,\infty}$ and $\boldsymbol{u}_*$ can be found from the velocity $\boldsymbol{u}_t(n_{\mathrm{ref}})$ at some reference height n_{ref} (with corresponding non-dimensional value ζ_{ref}), and the fact that

$$\boldsymbol{u}_t(n_0) = \boldsymbol{u}_{t,i} = \boldsymbol{u}_i \cdot \hat{\boldsymbol{t}} \approx 0, \tag{6}$$

where n_0 is the roughness length scale with corresponding non-dimensional value ζ_0:

$$\begin{aligned} \hat{\boldsymbol{u}}(\zeta_{\mathrm{ref}}) &= \eta_* \frac{\boldsymbol{u}_t(n_{\mathrm{ref}}) - \boldsymbol{u}_{t,\infty}}{\boldsymbol{u}_*} \\ &= (-i\chi\delta) e^{\delta\zeta_{\mathrm{ref}}} \end{aligned} \tag{7}$$

$$\begin{aligned} \hat{\boldsymbol{u}}_0 &= \eta_* \frac{\boldsymbol{u}_t(n_0) - \boldsymbol{u}_{t,\infty}}{\boldsymbol{u}_*} \\ &= -\eta_* \frac{\boldsymbol{u}_{t,\infty} - \boldsymbol{u}_{t,i}}{\boldsymbol{u}_*} \\ &= \left\{ -i\chi\delta e^{-\delta\xi_N} - \frac{\eta_*}{k}\left[\ln\frac{|\zeta_0|}{\xi_N} + (\delta - a)\xi_N + \frac{a}{2}\delta\xi_N^2) \right] \right\}, \end{aligned} \tag{8}$$

where we take the outer solution for $\hat{\boldsymbol{u}}$ regardless of the value of ζ_{ref} in Eq. (7), and where we have assumed that $|\zeta_0| \ll \xi_N$ in Eq. (8). Wall roughness is more commonly expressed in terms of a drag coefficient C_D, rather than as a roughness length scale. Eq. (8) is dominated by the term involving ζ_0 as long as buoyancy effects are negligible; this is the only term considered for the so-called quadratic drag formulation. Under these assumptions, the scalar version of Eq. (8) reduces to

$$u_{t,\infty} - u_{t,i} \approx -u_* \frac{1}{k} \ln\frac{|\zeta_0|}{\xi_N}. \tag{9}$$

Comparison with the quadratic drag relation,

$$u_*^2 \equiv C_D (u_{t,\infty} - u_{t,i})^2, \tag{10}$$

allows us to relate ζ_0 to C_D:

$$C_D = \left(-\frac{1}{k} \ln\frac{|\zeta_0|}{\xi_N} \right)^{-2}, \tag{11}$$

$$\zeta_0 = -\xi_N e^{-k/\sqrt{C_D}}. \tag{12}$$

3.1. Iterative algorithm

Given values for n_{ref} and $\boldsymbol{u}_{t,\mathrm{ref}} = \boldsymbol{u}_t(n_{\mathrm{ref}})$ (and assuming fixed η_* and a), we need an algorithm for finding $\boldsymbol{u}_{t,\infty}$ and $\boldsymbol{u}_*$. A successful iterative algorithm that converges quickly (typically in less than ten iterations) is the following:

$$
\begin{aligned}
&\hat{\boldsymbol{u}}_0 = \left\{-i\chi\delta e^{-\delta\xi_N} - \frac{\eta_*}{k}\left[\ln\frac{|\zeta_0|}{\xi_N} + (\delta - a)\xi_N + \frac{a}{2}\delta\xi_N^2)\right]\right\} \\
&\boldsymbol{u}_{t,\infty}^0 = \boldsymbol{u}_{t,\mathrm{ref}} \\
&\boldsymbol{u}_*^0 = -\eta_*\frac{\boldsymbol{u}_{t,\infty}^0 - \boldsymbol{u}_{t,i}}{\hat{\boldsymbol{u}}_0} \\
&\text{for } k = 1, 2, ...k_{\max} \\
&\qquad \zeta_{\mathrm{ref}}^{k-1} = \frac{n_{\mathrm{ref}} f}{\eta_* |\boldsymbol{u}_*^{k-1}|} \\
&\qquad \boldsymbol{u}_*^k = \frac{-\eta_*(u_{\mathrm{ref}} - \boldsymbol{u}_{t,i})}{\hat{\boldsymbol{u}}_0 + i\chi\delta e^{\delta\zeta_{\mathrm{ref}}^{k-1}}} \\
&\qquad \varepsilon = \frac{\left|\boldsymbol{u}_*^k - \boldsymbol{u}_*^{k-1}\right|}{\left|\boldsymbol{u}_*^k\right|} \\
&\qquad \text{if } \varepsilon < 10^{-6} \\
&\qquad\qquad \text{break} \\
&\qquad \text{end if} \\
&\text{end for} \\
&\boldsymbol{u}_{t,\infty} = -\frac{\boldsymbol{u}_*\hat{\boldsymbol{u}}_0}{\eta_*} + \boldsymbol{u}_{t,i}
\end{aligned}
$$

4. Coupled temperature and salinity boundary layer solution

The boundary layer structure of temperature and salinity are similar to those found in McPhee et al. (1987) and in Holland and Jenkins (1999). The so-called three equations, those for freezing temperature of sea water, heat flux and salt flux, must be simultaneously satisfied at the wall:

$$T_0 = a_T S_0 + b_T + c_T p_0(z_0), \tag{13}$$

$$Q_i^T - Q_o^T = Q_{\mathrm{latent}}^T, \tag{14}$$

$$Q_i^S - Q_o^S = Q_{\mathrm{brine}}^S, \tag{15}$$

where z_0 is the height (negative below sea level) of the interface. Subscripts i and o represent ice and ocean properties, respectively, while subscript 0 represents quantities at the interface. Equation (13) is a linearization of the freezing point valid for salinity in the range 4–40 psu. Equation (14) can be expanded as

$$-\rho_i c_{p,i}\kappa_i^T \frac{(T_i - T_0)}{\Delta n_i} - \rho_o c_{p,o}\left\langle u_n' T' \right\rangle_0 = -\rho_o u_{n,\mathrm{melt}} L_f, \tag{16}$$

where ρ_i and ρ_o are the densities of ice and ocean water, respectively, at the interface, $c_{p,i}$ and $c_{p,o}$ are the specific heat capacities of ice and ocean water, respectively, L_f is the latent heat of fusion, and $u_{n,\mathrm{melt}}$ is the melt rate in the ocean (related to the ice melt rate by mass continuity, $\rho_o u_{n,\mathrm{melt}} = \rho_i u_{n,\mathrm{melt},i}$). I have assumed that the temperature flux of ice can be parameterized in terms of its molecular diffusivity, κ_i^T, and using some reference temperature $T_i = T_i(\Delta n_i)$ a distance Δn_i above the ice/ocean interface. Similarly, the equation for salt flux can be reduced to

$$-\rho_o \left\langle u_n' S' \right\rangle_0 = -\rho_o u_{n,\mathrm{melt}} S_0, \tag{17}$$

where I have assumed that the salinity of the ice shelf is zero for all time (and therefore that the salt flux into the ice is also zero). This assumption will not be valid when frazil ice forms under ice shelves, but this process is thought to occur predominantly outside the ocean boundary layer, with saline ice being driven upward toward the ice/ocean interface by buoyancy (Holland and Jenkins, 1999). Therefore, the process of ice formation with nonzero salinity will not be considered within the boundary layer formulation.

Following McPhee et al. (1987), we can express the Reynolds averaged turbulent heat and salinity fluxes in terms of diffusion of bulk temperature and salinity normal to the interface,

$$\left\langle u_n' T' \right\rangle_0 = -\frac{\rho_i}{\rho_o} \frac{c_{p,i}}{c_{p,o}} \frac{\kappa_i^T}{\Delta n_i} (T_i - T_0) + u_{n,\mathrm{melt}} \frac{L_f}{c_{p,o}} = -\left(K_T \frac{\partial T}{\partial n} \right)_{n=0}, \tag{18}$$

$$\left\langle u_n' S' \right\rangle_0 = u_{n,\mathrm{melt}} S_0 = -\left(K_S \frac{\partial S}{\partial n} \right)_{n=0}, \tag{19}$$

where K_T and K_S are the turbulent plus molecular diffusivities for heat and salinity, respectively, analogous to the eddy viscosity commonly used in closures for the Reynolds averaged momentum equation. Turbulent fluxes of T and S can be generalized from Eqss (18) and (19) to arbitrary normal distance n from the surface:

$$\left\langle u_n' \lambda' \right\rangle = -K_\lambda \frac{\partial \lambda}{\partial n}, \tag{20}$$

where λ is one of T or S. This equation can be non-dimensionalized and then integrated from the interface to n to obtain

$$\frac{\lambda(n) - \lambda_0}{\langle u_n' \lambda' \rangle_0 / u_*} = \Phi_\lambda(n) = \int_n^0 \frac{u_*}{K_\lambda} \frac{\langle u_n' \lambda' \rangle}{\langle u_n' \lambda' \rangle_0} dn'. \tag{21}$$

I will assume that $T(n)$ and $S(n)$ have known values T_{ref} and S_{ref} at some reference distance from the interface n_{ref}, as I did for the tangential velocity in Sec. 3, and that $\Phi_{T,S}$ are known at this same reference distance (see below). With this assumption, I eliminate the unknown quantities T_0 and $u_{n,\mathrm{melt}}$ in Eqs (13) and (21) leaving an equation in a single unknown, S_0, the salinity at the interface:

$$
\begin{aligned}
0 &= c_2 S_0^2 + c_1 S_0 + c_0, && (22)\\
c_2 &= -a_T(d_1+1), && (23)\\
c_1 &= (T_{\mathrm{ref}} - b_T - c_T p_0) + d_1(T_i - b_T - c_T p_0) + d_2, && (24)\\
c_0 &= -d_2 S_{\mathrm{ref}}, && (25)\\
d_1 &= \Phi_{T,\mathrm{ref}} \frac{\rho_i}{\rho_o} \frac{c_{p,i}}{c_{p,o}} \frac{\kappa_i^T}{\Delta n_i u_*}, && (26)\\
d_2 &= \frac{\Phi_{T,\mathrm{ref}}}{\Phi_{S,\mathrm{ref}}} \frac{L_f}{c_{p,o}}. && (27)
\end{aligned}
$$

Equation (22) can be solved using the quadratic formula. If only one real, positive root exists, I take this to be the solution. Alternative methods for finding S_0 may be required to handle cases where Eq. (22) has either zero or two real, positive solutions. Once a solution for S_0 has been found, values for $u_{n,\mathrm{melt}}$ and T_0 can be computed from Eqs. (13), (19) and (21), with $\lambda \to S$ for the last equation

$$
\begin{aligned}
T_0 &= a_T S_0 + b_T + c_T p_0, && (28)\\
u_{n,\mathrm{melt}} &= u_* \frac{(S_{\mathrm{ref}} - S_0)}{\Phi_{S,\mathrm{ref}} S_0}. && (29)
\end{aligned}
$$

The remaining task is to specify the functional form for $\Phi_{S,T}(n)$. This is accomplished as in McPhee (1983) and McPhee et al. (1987) by assuming that the salinity and heat fluxes fall off linearly from their surface values to zero at the edge of the boundary layer

$$
\begin{aligned}
\langle u_n' \lambda' \rangle &= -K \frac{\partial \lambda}{\partial n} = \langle u_n' \lambda' \rangle_0 \left(1 - \frac{n}{n_{\mathrm{bl}}}\right), && (30)\\
n_{\mathrm{bl}} &= -k u_* \eta_* / f. && (31)
\end{aligned}
$$

The two papers differ slightly on how they assume the eddy viscosity (assumed to be the same as the turbulent diffusivity for both salinity and temperature) varies within the surface layer, and therefore how thick the surface layer is. McPhee et al. (1987) assumes the eddy viscosity is linear within the surface layer and constant within the outer layer

$$
\begin{aligned}
K &= \begin{cases} -knu_* & n_0 > n \geq n_{\mathrm{sl}}, \\ -kn_{\mathrm{sl}} u_* & n_{\mathrm{sl}} > n \geq n_{\mathrm{bl}}, \end{cases} && (32)\\
n_{\mathrm{sl}} &= -u_* \eta_*^2 \xi_N / f. && (33)
\end{aligned}
$$

With this definition, Eq. (30) can be integrated with respect to n.

$$
\begin{aligned}
\frac{\lambda(n) - \lambda_0}{\langle u_n' \lambda' \rangle_0 / u_*} =& \Phi_{\mathrm{turb}}(n) = \int_n^{n_0} \frac{u_*}{K_\lambda} \frac{\langle u_n' \lambda' \rangle}{\langle u_n' \lambda' \rangle_0} dn' \\
=& -\frac{1}{k} \begin{cases} \int_n^{n_0} \left(\frac{1}{n'} - \frac{1}{n_{\mathrm{bl}}}\right) dn' & n_0 > n \geq n_{\mathrm{sl}} \\ \int_{n_{\mathrm{sl}}}^{n_0} \left(\frac{1}{n'} - \frac{1}{n_{\mathrm{bl}}}\right) dn' + \frac{1}{n_{\mathrm{sl}}} \int_n^{n_{\mathrm{sl}}} \left(1 - \frac{n'}{n_{\mathrm{bl}}}\right) dn' & n_{\mathrm{sl}} > n > n_{\mathrm{bl}} \end{cases} \qquad (34)
\end{aligned}
$$

In the limit that $|n_0| \ll |n_{\rm sl}|$, this integral is

$$\Phi_{\rm turb}(n) = \frac{1}{k}\begin{cases} \ln\frac{n}{n_0} - \frac{n}{n_{\rm bl}} & n_0 > n \geq n_{\rm sl} \\ \ln\frac{n_{\rm sl}}{n_0} - \frac{n_{\rm sl}}{2n_{\rm bl}} - 1 + \frac{n}{n_{\rm sl}} - \frac{n^2}{2n_{\rm sl}n_{\rm bl}} & n_{\rm sl} > n > n_{\rm bl} \end{cases}. \tag{35}$$

McPhee et al. (1987) argues that molecular fluxes must also be taken into account

$$\Phi_{T,S}^{\rm mol} = b\left(\frac{u_* n_0}{\nu}\right)^{1/2}\left(\frac{\nu}{\kappa_o^{T,S}}\right)^{2/3}, \tag{36}$$

where ν is the viscosity of ocean water, κ_o^T and κ_o^S are the molecular diffusivities of temperature and salinity, respectively, and where $b = 1.57$ is a universal constant found by fit to observations. The total non-dimensional change in temperature and salinity from the surface to a distance n is the sum of Eqs. (35) and (36),

$$\Phi_{T,S}(n) = \Phi_{\rm turb}(n) + \Phi_{T,S}^{\rm mol}. \tag{37}$$

Since the buoyancy flux is directed vertically upward whereas the surface normal need not be vertical (though it will, in general, be close to vertical because of the large horizontal to vertical aspect ratio of the system), it is necessary to specify the relationship between $\langle u_n'\lambda'\rangle_0$ and $\langle u_z'\lambda'\rangle_0$ (where, again, $\lambda = \{T,S\}$), in order to compute the Obukhov length, Eq. (2). It seems reasonable to assume that the impact of buoyancy on the shape of the boundary layer normal to the interface will go to zero as the interface becomes vertical ($L \to \infty$, $\mu_* \to 0$, $\eta_* \to 1$ and $a \to 0$). Since

$$u_z' = \boldsymbol{u}' \cdot \hat{\boldsymbol{z}} = u_t'\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{z}} + u_n'\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}, \tag{38}$$

this suggests that $\langle u_t'\lambda'\rangle_0$ contributes negligibly to the vertical fluxes, so that the vertical flux of temperature and salinity is

$$\langle u_z'\lambda'\rangle_0 = \langle u_n'\lambda'\rangle_0 n_z, \tag{39}$$

where $n_z = \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}$ is the vertical component of the unit normal vector pointing from the ocean into the ice. Equation Eq. (2) becomes

$$L = \frac{\rho_o u_*^3}{gkn_z\left(\gamma_S\langle u_n'S'\rangle_0 - \gamma_T\langle u_n'T'\rangle_0\right)}, \tag{40}$$

where $\langle u_n'T'\rangle_0$ and $\langle u_n'S'\rangle_0$ are computed from Eq. (21), respectively.

Computation of $\boldsymbol{u}_*$ and $\boldsymbol{u}_\infty$ requires knowledge of L, and therefore of T_0 and S_0. Since computations of T_0 and S_0 themselves involve u_*, it is necessary to solve for all four of these parameters simultaneously using and iterative method, as proposed in McPhee et al. (1987). The iterative procedure begins by assuming neutral stability, giving values for $\boldsymbol{u}_*$ and $\boldsymbol{u}_\infty$ that do not depend on T_0 and S_0. Using this value for u_0, T_0 and S_0 can be computed. The results are used to compute a new value of L, μ_*, η_* and a. The process is repeated by obtaining new values of $\boldsymbol{u}_*$ and $\boldsymbol{u}_\infty$, and so on until a convergence criterion is met.

5. Interdependence of values at image points and boundary points

5.1. Tangential velocity

The picture becomes somewhat more complicated when the boundary layer solutions are applied within the IBM. We do not, in general, know the value of velocity, temperature or salinity at image points (the same as the values at n_{ref} in the previous sections) independently of their values at the boundary point of interest. This is because the value of a property at the image point is found by interpolation from neighboring values, including those at the boundary point. Since the value at the boundary point depends on the value at the image point, the we must simultaneously solve for both the image and boundary values.

The image value is found by interpolation (the linear sum of values at neighboring points times weights), which may include the boundary point. In what follows, we will assume the image point to be at a distance n from the boundary. (The boundary point is, of course a distance 0 from the boundary.) First, the decomposition of the velocity into normal and tangential components bears some exploration. The full velocity at the image point can be found via interpolation separately in the horizontal and vertical direcitons as follows:

$$\begin{aligned}\boldsymbol{u}^R(n) =&\hat{\boldsymbol{x}}\left(\sum_{j=1}^{J-1} w_{x,j}u_{x,j} + w_{x,J}u_x(0)\right) + \hat{\boldsymbol{y}}\left(\sum_{k=1}^{K-1} w_{y,k}u_{y,k} + w_{y,K}u_y(0)\right)\\ &+\hat{\boldsymbol{z}}\left(\sum_{l=1}^{L-1} w_{z,l}u_{z,l} + w_{z,L}u_z(0)\right), \qquad (41)\\ =&\tilde{\boldsymbol{u}}^R + \boldsymbol{W}^R\boldsymbol{u}^R(0), \qquad (42)\end{aligned}$$

$$\tilde{\boldsymbol{u}}^R \equiv \hat{\boldsymbol{x}}\left(\sum_{j=1}^{J-1} w_{x,j}u_{x,j}\right) + \hat{\boldsymbol{y}}\left(\sum_{k=1}^{K-1} w_{y,k}u_{y,k}\right) + \hat{\boldsymbol{z}}\left(\sum_{l=1}^{L-1} w_{z,l}u_{z,l}\right), \qquad (43)$$

$$\boldsymbol{W}^R \equiv \begin{bmatrix} w_{x,J} & 0 & 0 \\ 0 & w_{y,K} & 0 \\ 0 & 0 & w_{z,L}\end{bmatrix}, \qquad (44)$$

where the sums are over real fluid points, where ws are the interpolation weights, and where $w_{x,J}$, $w_{y,K}$ and/or $w_{z,L}$ may be zero if the boundary point is not included in the interpolation scheme for the image point velocity value. Superscripts R indicate that the velocities and the weighting matrix are expressed in the "regular" coordinate system $\{x,y,z\}$, rather than the "tilted" coordinate system defined below. The velocity at n has been expressed in terms of the known contribution $\tilde{\boldsymbol{u}}$ and the unknown contribution from $\boldsymbol{u}(0)$.

We define the tilted coordinate system aligned with the interface in terms of unit vectors $\hat{\boldsymbol{t}}$, $\hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{n}}$ (analogous to $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$, respectively), with $\hat{\boldsymbol{n}}$ normal to the surface, pointing from the ocean into the ice. We define the unit vectors tangent to the interface as

$$\hat{\boldsymbol{t}} \equiv \frac{\tilde{\boldsymbol{u}}_t}{|\tilde{\boldsymbol{u}}_t|}, \qquad (45)$$

$$\hat{\boldsymbol{s}} \equiv -\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}, \qquad (46)$$

where

$$\tilde{\boldsymbol{u}}_t = \tilde{\boldsymbol{u}} - \tilde{u}_n \hat{\boldsymbol{n}}, \tag{47}$$

$$\tilde{u}_n = \tilde{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}. \tag{48}$$

We can define a matrix $\boldsymbol{R}$ that rotates a vector from the regular to the tilted coordinates:

$$\boldsymbol{R} \equiv \begin{bmatrix} t_x & t_y & t_z \\ s_x & s_y & s_z \\ n_x & n_y & n_z \end{bmatrix}. \tag{49}$$

Then, the equivalent of Eq. (42) in the tilted coordinate frame is

$$\boldsymbol{u}(n) = \tilde{\boldsymbol{u}} + \boldsymbol{W}\boldsymbol{u}(0), \tag{50}$$

$$= \boldsymbol{R}\boldsymbol{u}^R(n), \tag{51}$$

$$\boldsymbol{W} \equiv \boldsymbol{R}\boldsymbol{W}^R\boldsymbol{R}^{-1}, \tag{52}$$

$$\boldsymbol{R}^{-1} = \boldsymbol{R}^T. \tag{53}$$

Equation (50) relates the three unknown velocity components at n to the three velocity components at 0. The two tangential velocity components at the interface are unknown (since we assume the ocean slips relative to the ice when the surface layer is not resolved). The normal velocity at the interface,

$$u_n(0) = u_{n,i}, \tag{54}$$

is known from the ice sheet model and the "no penetration" normal boundary condition. We split Eq. (50) into its tangent and normal components, making use of Eq. (54),

$$u_n(n) = \tilde{u}_n + w_{nt}u_n(0) + w_{ns}u_s(0) + w_{nn}u_{n,i}, \tag{55}$$

$$\boldsymbol{u}_t(n) = \tilde{\boldsymbol{u}}_t + \begin{bmatrix} w_{tn} \\ w_{sn} \end{bmatrix} u_{n,i} + \begin{bmatrix} w_{tt} & w_{ts} \\ w_{st} & w_{ss} \end{bmatrix} \boldsymbol{u}_t(0), \tag{56}$$

$$= \boldsymbol{u}_t^\dagger + \boldsymbol{W}^\dagger \boldsymbol{u}_t(0), \tag{57}$$

$$\boldsymbol{u}_t^\dagger \equiv \tilde{\boldsymbol{u}}_t + \begin{bmatrix} w_{tn} \\ w_{sn} \end{bmatrix} u_{n,i}, \tag{58}$$

$$\boldsymbol{W}^\dagger \equiv \begin{bmatrix} w_{tt} & w_{ts} \\ w_{st} & w_{ss} \end{bmatrix}, \tag{59}$$

where the w_{ij}'s are the components of $\boldsymbol{W}$ (in the tilted coordinates). Once $\boldsymbol{u}_t(0)$ is known (as described below), Eq. (55) can be used to find the normal velocity at the image point.

Making use of Eqs. (7) and (8), where the former is evaluated at $n_{\text{ref}} = n$ and $n_{\text{ref}} = 0$ (but assuming that the outer layer solution continues all the way to the boundary so that we avoid the very rapid change in velocity associated with the surface layer solution), we have

$$\boldsymbol{u}_{t,\infty} = -\frac{\hat{\boldsymbol{u}}_0 \boldsymbol{u}_*}{\eta_*} + \boldsymbol{u}_{t,i}, \tag{60}$$

$$\boldsymbol{u}_t(n) = \boldsymbol{u}_{t,\infty} - \frac{\boldsymbol{u}_* i\chi\delta}{\eta_*} e^{\delta n f/|u_*|\eta_*}, \tag{61}$$

$$\boldsymbol{u}_t(0) = \boldsymbol{u}_{t,\infty} - \frac{\boldsymbol{u}_* i\chi\delta}{\eta_*}. \tag{62}$$

Equations (57), (60), (61) and (62) can be solved simultaneously using an iterative scheme similar to the one in Sec. 3.1. The term used to compute $\boldsymbol{u}_*^k$ in terms of ζ_{ref}^{k-1} is replaced by

$$\boldsymbol{u}_*^k = -\eta_* \boldsymbol{M}^{-1}(\boldsymbol{u}_t^\dagger - \boldsymbol{u}_{t,i}), \tag{63}$$

$$\boldsymbol{M} \equiv \left[\hat{\boldsymbol{u}}_0 + i\chi\delta e^{\delta\zeta_{\text{ref}}}\right] - \boldsymbol{W}^\dagger \left[\hat{\boldsymbol{u}}_0 + i\chi\delta\right], \tag{64}$$

where complex numbers in $\boldsymbol{M}$ must be replaced by matrices as follows:

$$a + ib \rightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{65}$$

Note that, if $\boldsymbol{W} = \boldsymbol{0}$, then $\boldsymbol{u}_t(n) = \boldsymbol{u}_t^\dagger(n) = \boldsymbol{u}_{t,\text{ref}}$, and we recover the same term as in Sec. 3.1. In general, $|\boldsymbol{W}| \ll 1$, so that the terms involving $\boldsymbol{W}$ will be small perturbations to the earlier scheme, not expected to affect its convergence properties.

We compute the total velocity at a given ghost point using linear extrapolation from the boundary and image values,

$$\boldsymbol{u}(-n) = 2\boldsymbol{u}(0) - \boldsymbol{u}(n). \tag{66}$$

The normal component of velocity at the interface is supplied by the ice sheet model. The total fluid velocity at the interface is

$$\boldsymbol{u}(0) = u_t(0)\hat{\boldsymbol{t}} + u_s(0)\hat{\boldsymbol{s}} + u_{n,i}\hat{\boldsymbol{n}}. \tag{67}$$

The value of $\boldsymbol{u}(n)$ can be found from Eq. (50), now that all terms in this equation are known. From Eq. (66), the horizontal and vertical velocity components of the velocity can be computed for ghost points on the U and W grids, respectively.

5.2. *Normal velocity, temperature and salinity*

Temperature and salinity at the image point are related to their respective values at the boundary point by the relations

$$\lambda(n) = \lambda^\dagger(n) + w_N \lambda_0, \tag{68}$$

$$\lambda^\dagger(n) \equiv \sum_{j=1}^{N-1} w_j \lambda_j, \tag{69}$$

where the sums are over nearby neighbors that are in the fluid. Incorporating Eq. (68) into the solutions for temperature and salinity from Sec. 4 is a bit messier, but we will see that the result is still a quadratic equation for S_0, the salinity at the interface, whose solution can be used to compute the melt rate, interface temperature, and temperatures and salinities at the image and boundary points. These equations, together with Eq. (13) for the freezing temperature and Eq. (21) evaluated at $n_{\mathrm{ref}} = n$ lead to five total equations in the five unknowns S_0, T_0, $u_{n,\mathrm{melt}}$, $T(n)$, and $S(n)$. The five equations can be written as follows

$$S(n) = c_0 + c_1 S_0, \tag{70}$$
$$T(n) = c_2 + c_3 T_0, \tag{71}$$
$$T(n) = c_4 + c_5 T_0 + c_6 u_{n,\mathrm{melt}}, \tag{72}$$
$$T_0 = c_7 + c_8 S_0, \tag{73}$$
$$S(n) = c_9 S_0 + c_{10} u_{n,\mathrm{melt}} S_0, \tag{74}$$

where the constants c_n are given by

$$c_0 = \sum_{j=1}^{N-1} w_j S_j, \tag{75}$$
$$c_1 = w_N, \tag{76}$$
$$c_2 = \sum_{j=1}^{N-1} w_j T_j, \tag{77}$$
$$c_3 = w_N, \tag{78}$$
$$c_4 = b_0 \Phi_T(n) T_i, \tag{79}$$
$$c_5 = 1 - b_0 \Phi_T(n), \tag{80}$$
$$c_6 = \frac{L_f}{u_* c_{p,o}} \Phi_T(n), \tag{81}$$
$$c_7 = b_T + c_T p_0, \tag{82}$$
$$c_8 = a_T, \tag{83}$$
$$c_9 = 1, \tag{84}$$
$$c_{10} = \frac{\Phi_S(n)}{u_*}, \tag{85}$$

and where

$$b_0 = \frac{\rho_i}{\rho_o} \frac{c_{p,i}}{c_{p,o}} \frac{\kappa_i^T}{\Delta n_i u_*}. \tag{86}$$

All unknowns except for S_0 can be eliminated from Eqs. (70)–(74):

$$0 = e_0 + e_1 S_0 + e_2 S_0^2, \tag{87}$$

where

$$e_0 = d_2 d_3, \tag{88}$$
$$e_1 = d_2 d_4 - d_0 d_5, \tag{89}$$
$$e_2 = -d_1 d_5, \tag{90}$$
$$d_0 = c_2 + c_3 c_7 - c_4 - c_5 c_7, \tag{91}$$
$$d_1 = c_3 c_8 - c_5 c_8, \tag{92}$$
$$d_2 = -c_6, \tag{93}$$
$$d_3 = c_0, \tag{94}$$
$$d_4 = c_1 - c_9, \tag{95}$$
$$d_5 = -c_{10}. \tag{96}$$

We solve Eq. (87) using the quadratic formula. If the roots are complex or negative, an alternative formulation of the problem is required. If one root is positive while the other is not, the positive root is the physical solution. For the time being, if both roots are positive, the smaller root is assumed to be the physically real root. (We need a more sophisticated method that takes into account whether melting or freezing is occurring!). Given S_0, we find the melt rate using the relation

$$u_{n,\mathrm{melt}} = \frac{-d_0 - d_1 S_0}{d_2}, \tag{97}$$

T_0 using Eq. (73), $T(n)$ using Eq. (72) and $S(n)$ using Eq. (74).

5.3. The barotropic momentum

For this section, we use the notation of the POP reference manual, Smith and Gent (2002), where $\boldsymbol{u}$ is the horizontal velocity, w is the vertical velocity, ∇ is the horizontal gradient, etc.

A complication of imposing the full velocity at ghost points is that the velocity must remain divergence free. Since the geometry of the interface is specified using a height field (and, therefore, there is only one boundary intersection per vertical column), we can use the barotropic momentum and continuity equations to insure that the flow simultaneously remains divergence free and satisfies the boundary conditions on the vertical velocity. This can be accomplished by solving the rigid lid barotropic equations only in the region of "real" flow, between $z = -b$ and $z = -H$. The modified version of Eq. (126) from Smith and Gent (2002), the elliptic equation for the sea surface height η^{n+1} (where the sea surface height can be related to the surface pressure by $p_s = \rho_o g \eta$) is given by

$$\nabla \cdot (H - b) \nabla \eta^{n+1} = \nabla \cdot (H - b) \left[\frac{\hat{\boldsymbol{U}}}{g\alpha\tau} + \nabla \eta^{n-1} \right] + \frac{w_b}{g\alpha\tau}, \tag{98}$$

where $\hat{\boldsymbol{U}}$ is the auxiliary velocity as defined in Eq. (124) of Smith and Gent (2002), $\alpha = 1/3$, $\tau = 2\Delta t$ is twice the time step, and where the vertical velocity at the interface w_b is given by

$$w_b = \boldsymbol{u}(x, y, z = -b(x, y)) \cdot \hat{\boldsymbol{z}}, \tag{99}$$

which is computed by linear interpolated from the ocean grid cells above and below $z = -b$. This should reproduce $w_b = u_{n,i}$ to a good approximation when the interface is oriented horizontally.

There are three different integrals used to convert between barotropic and baroclinic momentum formulations. These are the vertical integral of the baroclinic velocity (which is required to be zero), the vertical integral of the explicitly treated forcing terms, and the integral used to compute the vertical component of the velocity. These first two integrals are performed as usual, except that the horizontal velocity and forcing are multiplied by a mask. Equation (97) from Smith and Gent (2002) becomes

$$\tilde{\boldsymbol{u}}'_k = \boldsymbol{u}'_k - \frac{1}{H_U} \sum_{k'=1}^{km} m_{U,k'} dz_{k'} \boldsymbol{u}'_{k'}, \tag{100}$$

$$m_{U,k} = \frac{H_U}{H_U - b_U} \begin{cases} 0 & k < k_b \\ \delta_U & k = k_b \\ 1 & k > k_b \end{cases} \tag{101}$$

$$z_{k_b - \frac{1}{2}} \geq -b_U \geq z_{k_b + \frac{1}{2}}, \tag{102}$$

$$\delta_U = \frac{-b_U - z_{k_b + \frac{1}{2}}}{\Delta z_{k_b}}. \tag{103}$$

Similarly, the barotropic forcing term is computed by

$$\boldsymbol{F}_B = \frac{1}{H_U} \sum_{k=1}^{km} m_{U,k} dz_k \boldsymbol{F}_k. \tag{104}$$

The vertical velocity is computed as usual by integrating the continuity equation downward starting with the (fictitious) vertical surface velocity w_0. In order to obtain the correct vertical velocity at $z = -b$, we must compute w_0 by integrating the continuity equation upward from $z = -b$ to the surface as folows:

$$w_{k_b - \frac{1}{2}} = w_b - \Delta z_{k_b} \nabla \cdot \left[(1 - \delta_U) \boldsymbol{u}_{,k_b} \right], \tag{105}$$

$$w_{k - \frac{1}{2}} = w_{k + \frac{1}{2}} - \Delta z_k \nabla \cdot \boldsymbol{u}_k \quad k = k_b - 1, \ldots 3, 2, \tag{106}$$

$$w_0 = w_{\frac{3}{2}} - \Delta z_1 \nabla \cdot \boldsymbol{u}_1. \tag{107}$$

Some of the δ_U's may be greater than one, in which case the interface intersects more than one cell in the vertical column, and continuity should be considered separately in each cell.

6. Boundary condition for underresolved boundary layers

Complications arise when the boundary layer is not resolved (which is nearly always the case, given the default spacing between vertical layers in POP. If the property we would like to faithfully reproduce at the boundary is the outer boundary layer solution, evaluated at the boundary then the scheme from Sec. 5.1 is adequate. On the other hand, if we would like the stress due to surface drag to be correct, we can assign a "partial slip" surface velocity such that the stress tensor computed in POP has the desired value at the surface (at least in the limit where the surface is oriented vertically). Vertical shear stresses in POP are approximated by

$$\tau_{x,z} = \nu_{\text{vert}} \frac{\partial u_x}{\partial z}, \tag{108}$$

$$\tau_{y,z} = \nu_{\text{vert}} \frac{\partial u_y}{\partial z}, \tag{109}$$

$$\boldsymbol{\tau}_z \approx \nu_{\text{vert}} \frac{\boldsymbol{u}_k - \boldsymbol{u}_{k+1}}{\Delta z_{k+\frac{1}{2}}}, \tag{110}$$

where ν_{vert} may be a function of horizontal or all three dimensions, and is computed using one of three available vertical mixing schemes, and where k is the vertical index and $\Delta z_{k+\frac{1}{2}}$ is the vertical spacing between U grid points. The boundary layer formulation in the previous sections defines the shear stress due to tangential flow at the interface by

$$\boldsymbol{\tau}_n = |\boldsymbol{u}_*| \boldsymbol{u}_*. \tag{111}$$

Since the normal is nearly vertical, assume that it is not too bad an approximation to equate Eqs. (110) and (111), with the finite differences now taken between the image and boundary point values. (We replace $\boldsymbol{u}_k$ by $\boldsymbol{u}_t(0)$, $\boldsymbol{u}_{k+1}$ by $\boldsymbol{u}_t(n)$ and $\Delta z_{k+\frac{1}{2}}$ by $|n|$.)

$$\nu_{\text{vert}} \frac{\boldsymbol{u}_t(n) - \boldsymbol{u}_t(0)}{n} = |\boldsymbol{u}_*| \boldsymbol{u}_*. \tag{112}$$

This relation replaces Eq. (62), relating $\boldsymbol{u}_t(0)$, $\boldsymbol{u}_t(n)$ and $\boldsymbol{u}_*$. (In the limit that $n \to 0$ and when $\nu_{\text{vert}} = \xi_N k u_*^2 \eta_*^2 / f$, the eddy viscosity as defined in the outer boundary layer, Eq. (112) is satisfied by the analytic solution for $\boldsymbol{u}_t(0)$ given by Eq. (62).) Solving Eqs. (57), (60), (61) and (112) for $\boldsymbol{u}_*$, we modify the iterative scheme from Sec. 3.1 with the following expression for $\boldsymbol{u}_*^k$:

$$\boldsymbol{u}_*^k = -\boldsymbol{M}^{-1} (\boldsymbol{u}_t^\dagger(n) - \boldsymbol{u}_{t,i}), \tag{113}$$

$$\boldsymbol{M} \equiv \frac{1}{\eta_*} \left(\boldsymbol{I} - \boldsymbol{W}^\dagger \right) \left[\hat{u}_0 + i\chi\delta e^{\delta \zeta_{\text{ref}}^{k-1}} \right] - \frac{n \left| \boldsymbol{u}_*^{k-1} \right|}{\nu_{\text{vert}}} \boldsymbol{W}^\dagger. \tag{114}$$

In the future, it may be worthwhile to take into account the fact that the boundary is not vertically aligned, so that the normal shear of the tangential velocity $\boldsymbol{\tau}_n$ is not the same as the vertical shear of the horizontal velocity $\boldsymbol{\tau}_z$.

A similar approach is required for computing $T(0)$ and $S(0)$, which are, in general, not equal to T_0 and S_0, respectively. POP represents (turbulent) vertical diffusion of a tracer λ by

$$\left\langle u_z' \lambda' \right\rangle_0 \approx -\kappa_{\text{vert}} \frac{\lambda_k - \lambda_{k+1}}{\Delta z_{k+\frac{1}{2}}}, \tag{115}$$

$$\approx -\kappa_{\text{vert}} \frac{\lambda(n) - \lambda(0)}{n} \tag{116}$$

where the tracer flux is given by an equation analogous to Eq. (21),

$$\frac{\lambda(n)-\lambda_0}{\langle u_n'\lambda'\rangle_0/u_*}=\Phi_\lambda(n). \tag{117}$$

By eliminating $\langle u_n'\lambda'\rangle_0$, we have

$$\lambda(0) =\lambda(n)-\kappa^\dagger(\lambda(n)-\lambda_0), \tag{118}$$

$$\kappa_\lambda^\dagger \equiv\frac{|n|u_*}{\kappa_{\text{vert}}\Phi_\lambda(n)}. \tag{119}$$

Note that $\lambda(0)=\lambda_0$ if $\kappa^\dagger=1$. The algorithm from Sec. 5.2 for computing boundary values only requires slight modification, where interpolation now makes use of $\lambda(0)$ instead of λ_0

$$\lambda(n) =\lambda^\dagger(n)+w_N\lambda(0), \tag{120}$$

By eliminating $\lambda(0)$ in Eqs. (118) and (120), we find that the algorithm form Sec. 5.2 only need be modified by replacing values for c_0 to c_3 by

$$c_0^\dagger =\frac{c_0}{1-w_N(1-\kappa_S^\dagger)}, \tag{121}$$

$$c_1^\dagger =\frac{c_1\kappa_S^\dagger}{1-w_N(1-\kappa_S^\dagger)}, \tag{122}$$

$$c_2^\dagger =\frac{c_2}{1-w_N(1-\kappa_T^\dagger)}, \tag{123}$$

$$c_3^\dagger =\frac{c_3\kappa_T^\dagger}{1-w_N(1-\kappa_T^\dagger)}. \tag{124}$$

Then, Eq. (30) is used to find $\lambda(0)$.

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